Maximal Sizes of Weak $(2,1)$-Sum-Free Sets in Finite Abelian Groups

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Abstract

The finite abelian group G is type I if $|G|$ has a prime divisor congruent to $2 \bmod 3$, type II if $|G|$ is divisible by 3 but has no divisors congruent to $2 \bmod 3$, and type III if all divisors of $|G|$ are congruent to $1 \bmod 3$. A subset $A \subset G$ is weakly $(2,1)$-sum-free if the set of all sums of 2 distinct elements of A is disjoint from A. We are interested in finding the size $\mu\hat{}(G,\{2,1\})$ of the largest weak $(2,1)$-sum-free subset of G. Here, we determine $\mu\hat{}(G,\{2,1\})$ for G of type I and some G of type II. We also present new constructions for weak $(2,1)$-sum-free sets for G of type III, and so find a new lower bound for $\mu\hat{}(G,\{2,1\})$.

1 Introduction

Suppose that $A = \{a_1, a_2, \ldots, a_m\}$ is a subset of a finite abelian group G, with $m \in \mathbb{N}$. Let h be a non-negative integer.

We will write hA for the (ordinary) h-fold sumset of A, which consists of sums of exactly h (not necessarily distinct) terms of A. More formally,

$$hA = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \mathbb{N}_0, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

For positive integers $k > l$, a subset A of a given finite abelian group G is called (k,l)-sum-free when

$$kA \cap lA = \emptyset.$$

We denote the maximum size of a (k,l)-sum-free subset of G as $\mu(G, \{k,l\})$. That is,

$$\mu(G, \{k,l\}) = \max\{|A| \mid A \subseteq G, (kA) \cap (lA) = \emptyset\}.$$

Similarly, we will write $h\hat{}A$ for the *restricted* h-fold sumset of A, which consists of sums of exactly h *distinct* terms of A:

$$h\hat{}A = \left\{ \sum_{i=1}^{m} \lambda_i a_i \;\middle|\; \lambda_1, \ldots, \lambda_m \in \{0,1\}, \sum_{i=1}^{m} \lambda_i = h \right\}.$$

For positive integers $k > l$, a subset A of a given finite abelian group G is called weakly (k,l)-sum-free when

$$k\hat{}A \cap l\hat{}A = \emptyset.$$

We denote the maximum size of a weak (k,l)-sum-free subset of G as $\mu\hat{}(G, \{k,l\})$. That is,

$$\mu\hat{}(G, \{k,l\}) = \max\{|A| \mid A \subseteq G, (k\hat{}A) \cap (l\hat{}A) = \emptyset\}.$$

2 Established values and bounds for μ and $\mu\hat{}$

For any positive integer x, we define

$$v_1(x,3) = \begin{cases} \left(1 + \frac{1}{p}\right) \frac{x}{3} & \text{if } x \text{ has prime divisors congruent to 2 mod 3,} \\ & \text{and } p \text{ is the smallest such divisor;} \\ \left\lfloor \frac{x}{3} \right\rfloor & \text{otherwise.} \end{cases}$$

Interestingly, the value of $v_1(x,3)$ is intimately related to $(2,1)$-sum-free sets. In fact, it has been proven that the largest size $(2,1)$-sum-free set of the cyclic group of order n is $v_1(n,3)$:

Theorem 1 (Diananda and Yap; [2] (G.4)) *For all positive integers* n*, we have*

$$\mu(\mathbb{Z}_n, \{2,1\}) = v_1(n,3).$$

Definition 2 *The exponent of a group is the order of the largest factor in its invarient decomposition.*

The largest size of a $(2,1)$-sum-free subset a group is dependent on its exponent in a rather satisfying way:

Theorem 3 (Green and Ruzsa; [2] (G.18)) *Let* κ *be the exponent of* G*. Then*

$$\mu(G, \{2,1\}) = \mu(\mathbb{Z}_\kappa, \{2,1\}) \cdot \frac{n}{\kappa} = v_1(\kappa,3) \cdot \frac{n}{\kappa}.$$

At least it is clear that

$$\mu(G, \{2,1\}) \geq v_1(\kappa,3) \cdot \frac{n}{\kappa}.$$

We can write $G = G_1 \times \mathbb{Z}_\kappa$ with $|G_1| = \frac{n}{\kappa}$, and suppose that $A \subset \mathbb{Z}_\kappa$ is a $(2,1)$-sum-free set of maximal size. Then

$$|G_1 \times A| = \frac{n}{\kappa} \cdot v_1(\kappa,3)$$

and $G_1 \times A$ must be $(2,1)$-sum-free in G, for if not we would have some (g_1, a_1), (g_2, a_2), and (g_3, a_3) in $G_1 \times A$ for which

$$(g_1 + g_2, a_1 + a_2) = (g_1, a_1) + (g_2, a_2) = (g_3, a_3),$$

contradicting that A is $(2,1)$-sum free. Proving that $\mu(G, \{2,1\}) \leq v_1(\kappa,3) \cdot \frac{n}{\kappa}$ required extensive computational work, but by doing so, Green and Rusza finally finished a four-decade-long search in 2005.

It should be mentioned that μ is a lower bound of $\mu\hat{}$ and that a more general lower bound has been established:

Proposition 4 (Bajnok; [2] (G.63)) *Suppose that* G *is an abelian group of order* n *and exponent* κ*. Then, for all positive integers* k *and* l *with* $k > l$ *we have*

$$\mu\hat{}(G, \{k,l\}) \geq \mu(G, \{k,l\}) \geq v_{k-l}(\kappa, k+l) \cdot \frac{n}{\kappa}.$$

Zannier starts the work on considering weak $(2,1)$-sum-free subsets by proving that the maximal size of a weakly $(2,1)$-sum-free subset of a cyclic group $\mathbb{Z}_n$ is $v_1(n,3)$ if n has prime divisors congruent to 2 mod 3 and $v_1(n,3)+1$ otherwise.

Theorem 5 (Zannier; [2] (G.67)) *For all positive integers we have*

$$\mu\hat{}(\mathbb{Z}_n,\{2,1\}) = \begin{cases} \left(1+\frac{1}{p}\right)\frac{n}{3} & \text{if } n \text{ has prime divisors congruent to 2 mod 3,} \\ & \text{and } p \text{ is the smallest such divisor;} \\ \left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

We will see in the proofs of Theorems 10 and 11 that the techniques used in Zannier's proof can be extended to be used on noncylcic groups.

In my last paper, I began the effort on evaluating $\mu(G,\{2,1\})$ for noncyclic groups G. The following have been established and will be pertinent to our work here.

Proposition 6 (Francis; [3]) *For all groups G with order n, and for all positive integers $k > l$,*

$$\mu\hat{}(G,\{k,l\}) \leq \left\lfloor \frac{n-2+l+k}{2} \right\rfloor.$$

This upper bound allows to find $\mu(G,\{2,1\})$ for any even G with even $|G|$.

Proposition 7 (Francis; [3]) *For any G with $|G| = n \equiv 0 \bmod 2$,*

$$\mu\hat{}(G,\{2,1\}) = \frac{n}{2}.$$

Through specific constructions of weakly $(2,1)$-sum-free sets, we have the following.

Theorem 8 (Francis; [3]) *For any positive integer $w \equiv 1 \bmod 2$,*

$$\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w},\{2,1\}) \geq 3w+1.$$

Theorem 9 (Francis; [3]) *For all positive $\kappa \equiv 1 \bmod 6$,*

$$\mu\hat{}(\mathbb{Z}_\kappa^2,\{2,1\}) \geq \frac{\kappa-1}{3}\cdot\kappa+1.$$

3 Divide and Conquer

As it has been done before, we find it natural to categorize groups into three types. A group G is called type I if $|G|$ has a divisor congruent to 2 mod 3, type II if $|G|$ is divisible by 3 but has no prime divisors congruent to 2 mod 3, and type III if all divisors of $|G|$ are congruent to 1 mod 3.

3.1 Groups G of type I

We can completely determine $\mu\hat{}(G,\{2,1\})$ for G of type I.

Theorem 10 *If G is a group of type I, then*

$$\mu\hat{}(G,\{2,1\}) = \mu(G,\{2,1\}).$$

PROOF. It is clear that $\mu\hat{}(G,\{2,1\}) \geq \mu(G,\{2,1\})$, so we must show that for any weak $(2,1)$-sum-free $A \subseteq G$,

$$|A| \leq \mu(G,\{2,1\}).$$

Let $n = |G|$, κ be the exponent of G, and p be the smallest prime divisor of n that is congruent to 2 mod 3. Then Theorems 1 and 3 guarantee that

$$\mu(G,\{2,1\}) = \mu(\mathbb{Z}_\kappa,\{2,1\}) \cdot \frac{n}{\kappa} = v_1(\kappa,3) \cdot \frac{n}{\kappa} = \left(1+\frac{1}{p}\right)\frac{\kappa}{3}\cdot\frac{n}{\kappa} = \left(1+\frac{1}{p}\right)\frac{n}{3},$$

so to prove our claim it is sufficient to show that

$$|A| \leq \left(1+\frac{1}{p}\right)\frac{n}{3}.$$

Note that if $|G|$ is even, then 2 is the smallest prime congruent to 2 mod 3 that divides n, so using Proposition 7 we can write

$$\mu\hat{}(G,\{2,1\}) = \frac{n}{2} = \left(1+\frac{1}{2}\right)\cdot\frac{n}{3} = \mu(G,\{2,1\}).$$

Also, when G is cyclic, Theorems 1 and 5 give us

$$\mu\hat{}(G,\{2,1\}) = \mu\hat{}(\mathbb{Z}_n,\{2,1\}) = \left(1+\frac{1}{p}\right)\frac{n}{3} = v_1(n,3) = \mu(\mathbb{Z}_n,\{2,1\}) = \mu\hat{}(G,\{2,1\}).$$

We will continue and assume G is noncyclic and that 2 does not divide $|G|$.

If $A = \{0\}$, our claim holds trivially, so suppose that A contains some nonzero element a. In this case, A may not contain 0, for if $0 \in A$, then $0 + a = a \in A$ would contradict A being weakly $(2,1)$-sum-free. Therefore, we can assume that $0 \notin A$.

When A is $(2,1)$-sum-free, our claim holds, so assume that A is not $(2,1)$-sum-free. This means that there is some $a_0 \in A$ for which $2a_0 \in A$. Since $0 \notin A$, we have that $a_0 \neq 0$, so $a_0 \neq 2a_0$.

Let
$$A_1 = a_0 + (A \setminus \{a_0\}) = \{a_0 + a \mid a \in A \setminus \{a_0\}\}$$
and
$$A_2 = 2a_0 + (A \setminus \{a_0, 2a_0\}) = \{2a_0 + a \mid a \in A \setminus \{a_0, 2a_0\}\}.$$
Note that $A_1 \subseteq 2\hat{}A$ and $A_2 \subseteq 2\hat{}A$, so A is disjoint from both A_1 and A_2. Furthermore, A_1 and A_2 are disjoint too, since otherwise we would have elements $a_1 \in A \setminus \{a_0\}$ and $a_2 \in A \setminus \{a_0, 2a_0\}$ for which
$$a_0 + a_1 = 2a_0 + a_2,$$
but then
$$a_1 = a_0 + a_2,$$
contradicting that A and A_1 are disjoint. Since A, A_1, and A_2 are pairwise disjoint, we have
$$|A| + |A_1| + |A_2| = 3|A| - 3 \leq n.$$
We know that n must be at most $3p$: p divides n but $n \neq p$ since G is not cyclic and $n \neq 2p$ since n is odd. Therefore,
$$|A| \leq \left\lfloor \frac{n}{3} \right\rfloor + 1 \leq \frac{n}{3} + \frac{n}{3p} = \left(1 + \frac{1}{p}\right)\frac{n}{3},$$
as desired. □

3.2 Groups G of type II

Recall that a group G is called type II if $|G|$ is divisible by 3 but has no prime divisors congruent to 2 mod 3.

Theorem 11 *If G is a group of type II, then*

$$\mu\hat{}(G,\{2,1\}) \leq \mu(G,\{2,1\}) + 1.$$

PROOF. It is clear that $\mu\hat{}(G,\{2,1\}) \geq \mu(G,\{2,1\})$, so we must show that for any weak $(2,1)$-sum-free $A \subseteq G$,

$$|A| \leq \mu(G,\{2,1\}) + 1.$$

Theorems 1 and 3 guarantee that

$$\mu(G,\{2,1\}) + 1 = \mu(\mathbb{Z}_\kappa,\{2,1\}) \cdot \frac{n}{\kappa} + 1 = v_1(\kappa,3) \cdot \frac{n}{\kappa} + 1 = \left\lfloor \frac{\kappa}{3} \right\rfloor \cdot \frac{n}{\kappa} + 1,$$

where κ is the exponent of G. Since κ is divisible by the order of every cyclic group in the invarient factorization of G, and 3 is prime, since 3 divides $|G|$, 3 divides κ as well. This means that it is sufficient to show that

$$|A| \leq \left\lfloor \frac{\kappa}{3} \right\rfloor \cdot \frac{n}{\kappa} + 1 = \frac{\kappa}{3} \cdot \frac{n}{\kappa} + 1 = \frac{n}{3} + 1.$$

If $A = \{0\}$, our claim holds trivially, so suppose that A contains some nonzero element a. In this case, A may not contain 0, for if $0 \in A$, then $0 + a = a \in A$ would contradict A being weakly $(2,1)$-sum-free. Therefore, we can assume that $0 \notin A$.

When A is $(2,1)$-sum-free, our claim holds, so assume that A is not $(2,1)$-sum-free. This means that there is some $a_0 \in A$ for which $2a_0 \in A$. Since $0 \notin A$, we have that $a_0 \neq 0$, so $a_0 \neq 2a_0$.

Let

$$A_1 = a_0 + (A \setminus \{a_0\}) = \{a_0 + a \mid a \in A \setminus \{a_0\}\}$$

and

$$A_2 = 2a_0 + (A \setminus \{a_0, 2a_0\}) = \{2a_0 + a \mid \setminus \{a_0, 2a_0\}\}.$$

Note that $A_1 \subseteq 2\hat{}A$ and $A_2 \subseteq 2\hat{}A$, so A is disjoint from both A_1 and A_2. Furthermore, A_1 and A_2 are disjoint too, since otherwise we would have elements $a_1 \in A \setminus \{a_0\}$ and $a_2 \in A \setminus \{a_0, 2a_0\}$ for which

$$a_0 + a_1 = 2a_0 + a_2,$$

but then

$$a_1 = a_0 + a_2,$$

contradicting that A and A_1 are disjoint. Since A, A_1, and A_2 are pairwise disjoint, we have

$$|A| + |A_1| + |A_2| = 3|A| - 3 \leq n,$$

which implies that

$$|A| \leq \frac{n}{3} + 1,$$

as desired. □

Corollary 12 *If* $w \equiv 1 \bmod 2$ *has no prime divisors congruent to* $2 \bmod 3$*, then*

$$\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) = \mu(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) + 1 = 3w + 1.$$

PROOF. By Proposition 8

$$\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) \geq 3w + 1$$

and since $\mathbb{Z}_3 \times \mathbb{Z}_{3w}$ is type II, we can use Theorems 11 and 3 to write

$$\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) \leq \mu(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) + 1 = v_1(3w, 3) \cdot \frac{9w}{3w} + 1 = 3w + 1.$$

□

When w does have a prime divisor congruent to $2 \bmod 3$, it is still true that $\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) \geq 3w + 1$, however, this is a fairly poor lower bound and the actual value of $\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\})$ is found in the previous section.

It is computationally verified that

$$\mu\hat{}(\mathbb{Z}_3^3, \{2,1\}) = 9 = \mu(\mathbb{Z}_3^3, \{2,1\})$$

with

$$\begin{aligned} A = \{&(0,0,1), (0,1,0), (0,2,2), \\ &(1,0,0), (1,1,2), (1,2,1), \\ &(2,0,2), (2,1,1), (2,2,0)\} \end{aligned}$$

weakly $(2,1)$-sum-free in $\mathbb{Z}_3^3$. The value of $\mu\hat{}(G, \{2,1\})$ for larger groups G has thus far been too computationally intensive to find.

A more general categorization on the groups G of type II for which

$$\mu\hat{}(G, \{2,1\}) = \mu(G, \{2,1\}) + 1$$

is not known.

3.3 Groups G of type III

Recall that a group G is called type III if all divisors of $|G|$ are congruent to 1 mod 3. We will establish a lower bound for $\mu\hat{}(G,\{2,1\})$ using a similar method to that in Proposition 3.

Proposition 13 *Let* $G = G_1 \times \mathbb{Z}_\kappa$ *with* $|G_1|$ *odd and* $\kappa \equiv 1$ *mod* 6*. Define* $D \subset \mathbb{Z}_\kappa$ *as*

$$D = \left\{\pm 1, \pm 3, \ldots, \pm\frac{\kappa-4}{3}\right\},$$

and $A \subset G$ *as*

$$A = \left\{\left(0,\ldots,0,\frac{\kappa+2}{3}\right)\right\} \cup (G_1 \times D).$$

Then A *is weakly* $(2,1)$*-sum-free in* G*.*

PROOF. Since D is an arithmetic progression of common difference two, we can easily write

$$2D = \left\{0, \pm 2, \pm 4, \ldots, \pm 2 \cdot \frac{\kappa-4}{3}\right\},$$

another progression. Observe that the progression in $2D$ continues the progression in D, skipping the term $\frac{\kappa+2}{3}$:

$$\frac{\kappa-4}{3} + 2 = \frac{\kappa+2}{3}$$

and

$$\frac{\kappa+2}{3} + 2 = \frac{\kappa+8}{3} \equiv \frac{\kappa+8}{3} - \kappa = -2 \cdot \frac{\kappa-4}{3}.$$

Similarly, the progression in D continues the progression in $2D$, skipping the term $-\frac{\kappa+2}{3}$.

Furthermore, since κ is odd, the arithmetic progression will repeat in at least κ terms, and

$$\begin{aligned}
|2D| + |D| &= 1 + \frac{1}{2}\left(2\frac{\kappa-4}{3} - 2\frac{4-\kappa}{3}\right) + \frac{\kappa-1}{3} \\
&= 2\frac{\kappa-4}{3} + \frac{\kappa-1}{3} + 1 \\
&= \kappa - 2 \\
&< \kappa,
\end{aligned}$$

so D and $2D$ are disjoint. Thus, D is $(2,1)$-sum-free, and we can partition $\mathbb{Z}_\kappa$:

$$\mathbb{Z}_\kappa = D \cup \left\{\frac{\kappa+2}{3}\right\} \cup 2D \cup \left\{-\frac{\kappa+2}{3}\right\}.$$

Now, if we take any (not necessarily distinct)

$$a_1, a_2 \in A \setminus \left\{\left(0,\ldots,0,\frac{\kappa+2}{3}\right)\right\},$$

we know the last coordinates of a_1 and a_2 are in D, so the last coordinate of a_1+a_2 is in $2D$, not in $D \cup \{\frac{\kappa+2}{3}\}$; hence, $a_1+a_2 \notin A$. It remains to be shown is that

$$\left(0,\ldots,0,\frac{\kappa+2}{3}\right) + \left(A \setminus \left\{\left(0,\ldots,0,\frac{\kappa+2}{3}\right)\right\}\right)$$

is disjoint from A, for which it is sufficient to show that $D + \frac{\kappa+2}{3}$ is disjoint from D. Well,

$$D + \frac{\kappa+2}{3} = \left\{2,4,\ldots,2\cdot\frac{\kappa-4}{3}, 2\cdot\frac{\kappa-1}{3}\right\} \subset \left(2D \cup \left\{-\frac{\kappa+2}{3}\right\}\right),$$

which is disjoint from D, so we are done. □

Theorem 14 *For every group G of type III,*

$$\mu\hat{}(G,\{2,1\}) \geq \mu(G,\{2,1\}) + 1.$$

PROOF. First note that if G is type III, then all divisors of $|G|$ are congruent to 1 mod 3. Namely the exponent κ of G is congruent to 1 mod 3. Since $2 \not\equiv 1$ mod 3, $\kappa \not\equiv 4$ mod 6. Thus $\kappa \equiv 1$ mod 6. Then with notation as above, A is weakly $(2,1)$-sum-free in G, so

$$\begin{aligned}
\mu\hat{}(G,\{2,1\}) \geq |A| &= 1 + |G_1 \times D| \\
&= 1 + \frac{n}{\kappa}\cdot\frac{\kappa-1}{3} \\
&= 1 + \frac{n}{\kappa}\cdot\left\lfloor\frac{\kappa}{3}\right\rfloor \\
&= 1 + \frac{n}{\kappa}\cdot v_1(\kappa,3) \\
&= 1 + \mu(G,\{2,1\}).
\end{aligned}$$

□

4 Future work

As stated before, a more general categorization on the groups G of type II for which

$$\mu\hat{}(G,\{2,1\}) = \mu(G,\{2,1\}) + 1$$

is not known, and would be valuable to find. It is curious that the value of $\mu\hat{}(G,\{2,1\})$ did not depend on the exponent of the group G for type I, but seems to for type II.

It is also still open to prove or disprove that

$$\mu\hat{}(G,\{2,1\}) = \mu(G,\{2,1\}) + 1$$

for every group G of type III. This task presents to be very challenging.

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